Stabilization of the weak stationary solutions to 2D g - Navier - Stokes equations

Ổn định hóa nghiệm dừng yếu của hệ phương trình g - Navier - Stokes hai chiều

Nguyen Viet Tuan

Email: nguyentuandhsd@gmail.com

Sao Do University

Received date: 4/10/2020

Accepted date: 26/12/2020

Published date: 31/12/2020

Abstract

In this paper, we consider the g-Navier-Stokes equations in a two-dimensional bounded domain Ω_g. We stabilize an unstable weak stationary solution by using linear multiplicative Ito noise.

Keywords: *2D g-Navier-Stokes equations; stabilization; weak stationary solution.*

Tóm tắt

Trong bài báo này, chúng ta xét hệ phương trình g-Navier-Stokes trong miền hai chiều bị chặn Ω_g. Chúng ta ổn định hóa một nghiệm dừng yếu không ổn định bằng cách sử dụng một nhiễu Ito nhân tuyến tính.

Từ khóa: *Hệ phương trình g-Navier-Stokes hai chiều; ổn định hóa; nghiệm dừng yếu.*

1. INTRODUCTION

Let Ω_g be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\delta\Omega_g$. We consider the following 2D g-Navier-Stokes equations.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u.\nabla)u + \nabla p = f & \text{in } \Omega_g \times \mathbb{R}^+, \\ \nabla.(gu) = 0 & \text{in } \Omega_g \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega_g, \\ u(x,0) = u_0(x) & \text{in } \Omega_g, \end{cases} \quad (1)$$

Where:

$u = u(x,t) = u(u_1, u_2)$ is the unknown velocity vecto;

$p = p(x,t)$ is the unknown pressure;

$\nu > 0$ is the kinematic viscosity coefficient;

u_0 is the initial velocity.

The 2D g-Navier-Stokes equations arise in a natural way when we study the standard 3D Navier-Stokes problem in a 3D thin domain $T_g = \Omega_g \times (0,g)$ (see [9]). As mentioned in [9, 10], good properties of the 2D g-Navier-Stokes equations can lead to an initial study of the 3D Navier-Stokes equations in the thin domain T_g. In the last few years, the existence and long-time behavior of solutions in terms of existence of attractors for 2D g-Navier-Stokes equations have been studied extensively in both autonomous and non-autonomous cases (see e.g. [1, 2, 4, 5, 6, 7, 9, 12] and references therein). The stability and stabilization of strong stationary solutions to 2D g-Navier-Stokes equations by using an internal feedback control with support large enough were studied recently in [8].

In this paper, we continue studying the stabilization of weak stationary solutions to problem (1). To do this, we assume that the function g satisfies the following assumption:

(G) $g \in W^{1,\infty}(\Omega_g)$ such that $0 < m_0 \le g(x) \le M_0$ for all $x = (x_1, x_2) \in \Omega_g$ and $|\nabla g|_\infty < m_0 \lambda_1^{\frac{1}{2}}$, where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω_g (i.e. the operator A is defined in Section 2 below).

This paper is organized as follows. In Section 2, for convenience of the reader, we recall some results on function spaces and operators related to 2D g-Navier-Stokes equations which will be used. In Section 3, we show that any unstable weak

Reviewers: 1. Assoc. Prof. Dr. Khuat Van Ninh
2. Assoc. Prof. Dr. Nguyen Van Tuyen

stationary solution to 2D g -Navier-Stokes equations can be exponentially stabilized by feedback controll as a multiplicative white noise term.

2. Preliminaries

Let $L^2(\Omega_g,g) = (L_2(\Omega_g))^2$ and $H_0^1(\Omega_g,g) = (H_0^1(\Omega_g))^2$ be endowed, respectively, with the inner products.

$$(u,v)_g = \int_{\Omega_g} uvgdx,\quad u,v \in L^2(\Omega_g,g),$$

$$((u,v))_g = \sum_{i=1}^{2}\int_{\Omega_g} \nabla u_i.\nabla v_i gdx,$$

$u=(u_1,u_2), v=(v_1,v_2) \in H_0^1(\Omega_g,g)$ and norms $|u|^2=(u,u)_g, \|u\|^2=((u,u))_g$. Thanks to assumption (G) the norms $|.|$ and $\|.\|$ are equivalent to the usual ones in $(H_0^1(\Omega_g))^2$.

Let

$$\mathcal{V} = \left\{u \in \left(C_0^\infty(\Omega_g)\right)^2 : \nabla.(gu) = 0\right\}$$

Denote by H_g the closure of ν in $L^2(\Omega_g,g)$, and by V_g the closure of ν in $H_0^1(\Omega_g,g)$. It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $\|.\|_*$ for the norm in V_g', and $\langle .,. \rangle$ for duality pairing between V_g and V_g'.

We define the g- Stokes operator $A: V_g \to V_g'$ by $(Au,v) = ((u,v))_g$ for all $u,v \in V_g$.

Then $A = -P_g\Delta$ and $D(A) = H^2(\Omega_g,g) \cap Vg$, where P_g is the ortho-projector from $H_0^1(\Omega_g,g)$ onto V_g. We also define the operator $B: Vg \times Vg \to V_g'$ by $(B(u,v),w) = b(u,v,w)$, for all $u,v,w \in V_g$ where.

$$b(u,v,w) = \sum_{i,j=1}^{2}\int_{\Omega_g} u_i \frac{\partial v_j}{\partial x_i} w_j gdx.$$

It is easy to check that if $u,v,w \in V_g$, then $b(u,v,w) = -b(u,v,w)$, $b(u,v,v) = 0$.

We also set.

$B(u) = B(u,u)$ for all $u \in V_g$.

We recall some known results which will be used in the paper.

Lemma 2.1 ([1]). If $n=2$ then

$$|b(u,v,w)| = \begin{cases} c_1|u|^{\frac12}\|u\|^{\frac12}\|v\||w|^{\frac12}\|w\|^{\frac12}, \forall u,v,w \in V_g, \\ c_2|u|^{\frac12}\|u\|^{\frac12}\|v\|^{\frac12}|Av|^{\frac12}|w|^{\frac12}, \forall u \in V_g, \\ \qquad v \in D(A), w \in H_g, \\ c_3|u|^{\frac12}|Au|^{\frac12}\|v\||w|^{\frac12}, \forall u \in D(A), v \in V_g, \\ \qquad w \in H_g. \end{cases}$$

Where $c_i, i=1...3,$ are appropriate constants.

Lemma 2.2 ([3]) Let $u \in L^2(0,T,V_g)$ then the function Cu defined by:

$$(Cu(t),v)_g = \left(\left(\frac{\nabla g}{g}.\nabla\right)u,v\right)_g$$

$$= b\left(\frac{\nabla g}{g},u,v\right), \forall v \in V_g,$$

Belongs to $L^2(0,T,V_g)$ and hence also belongs to $L^2(0,T,V_g')$ Moreover,

$|Cu(t)| \leq \frac{|\nabla g|_\infty}{m_0}.\|u(t)\|$, for a.e. $t \in (0,T)$

And

$\|Cu(t)\|_* \leq \frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac12}}.\|u(t)\|$, for a.e. $t \in (0,T)$

We give the definition of the weak stationary solutions to 2D g -Navier-Stokes equations (1).

Definition 2.1. Let $f \in V_g'$ be given. A weak stationary solution to problem (1) is an element $u^* \in V_g$ such that.

$\nu Au^* + \nu Cu^* + B(u^*,u^*) = f$ in V_g'.

The following result was proved in [11].

Theorem 2.1. Let f be given in V_g'. Then,

(i) there exists a weak stationary solution $u^* \in V_g$ to (1);

(ii) furthermore, if the following condition holds.

$$\left[\nu\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac12}}\right)\right]^2 > \frac{c_1}{\lambda_1^{\frac12}}\|f\|_* \qquad (2)$$

Where:

c_1 is the constant in Lemma 2.1, then the weak stationary solution to (1) is unique and globally exponentially stable. That is, for any initial data $u_0 \in Hg$ and the any weak solution $u(t)$ of (1.1), then there exists $\lambda > 0$ such that.

$$|u(t)-u^*|^2 \leq |u_0-u^*|^2 e^{-\lambda t}, \forall t \geq 0.$$

Moreover, if u^* satisfying.

$$\|u^*\| \leq \frac{\nu}{4c}\lambda_1^{\frac12}\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac12}}\right),$$

$c = max\{c_1,c_2,c_3\}$,

$$|u_0-u^*| \leq k_0\nu\lambda_1^{\frac12}\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac12}}\right)$$

Where k_0 is a positive real number, then there exists k, $\alpha > 0$ such that.

$$\|u(t)-u^*\|^2 \leq k|u_0-u^*|^2 e^{-\alpha t}, \forall t \geq t^* > 0.$$

3. Stabilization by linear multiplicative ito noise

In this section, we will study the stabilization of the solution u^* by using a stochastic perturbation of the type $h(t,u)dW(t) = \sigma(u - u^*)dW(t)$.

We consider the following controlled 2D g-Navier-Stokes equations in perturbed by a linear multiplicative white noise.

$$\begin{cases} du = \left[\nu\Delta u - (u.\nabla)u - \nabla p + f(x)\right]dt \\ \qquad + \sigma\left(u-u^*\right)d\mathrm{W}(t) & \text{in } \Omega_g \times \mathbb{R}^+, \\ \nabla.(gu) = 0 & \text{in } \Omega_g \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega_g \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega_g, \end{cases} \tag{3}$$

Where σ is a real number, and

$W(t): \Omega_g \to \mathbb{R}, t \in \mathbb{R}$

is a one dimensional Wiener process defined on a probability space (Ω, P,F).

Thus, we can write (2) as follows in the abstract mathematical setting:

$$du = [-\nu Au - \nu Cu - B(u) + f]dt + \sigma(u - u^*)d\mathrm{W}(t) \text{ in } V_g^{'}. \tag{4}$$

It is noticed that the stationary solution u^* of problem (1) is also a solution to perturbed problem (3).

The following theorem is our main result in this section.

Theorem 3.1. Let f be given in $V_g^{'}$ and u^* be any weak stationary solution to (1) such that.

$$\|u^*\| \le \frac{\nu}{c_1}\lambda_1^{\frac{1}{2}}\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac{1}{2}}}\right) \tag{5}$$

Then the stationary solution u^* is almost sure exponentially stable. That is, there exists $\Omega_0 \subset \Omega, P(\Omega_0) = 0$ such that for $\omega \notin \Omega_0$ there exists $T(w) > 0$ such that any weak solution $u(t)$ to (3) following estimate holds.

$$|u(t) - u^*|^2 \le |u_0 - u^*|^2 e^{-\gamma t}, \forall t \ge T(\omega)$$

Where

$$\gamma = \frac{1}{2}\lambda_1\left[2\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac{1}{2}}}\right) - \frac{2c_1}{\lambda_1^{1/2}}\|u^*\| + \frac{\sigma^2}{\lambda_1}\right] > 0.$$

Proof. Let us apply Ito's formula for $\left|u(t)-u^*\right|^2$. Then it follows.

$$\begin{aligned}\left|u(t)-u^*\right|^2 &= \left|u(0)-u^*\right|^2 - 2\int_0^t\left\langle \nu Au(s), u(s)-u^*\right\rangle ds \\ &\quad - 2\int_0^t\left\langle \nu B(u(s)), u(s)-u^*\right\rangle ds \\ &\quad - 2\int_0^t\left(\nu Cu(s), u(s)-u^*\right)_g ds \\ &\quad + 2\int_0^t\left(f, u(s)-u^*\right)ds + \int_0^t\left|h(s,u(s))\right|^2 ds \\ &\quad + 2\int_0^t\left(u(s)-u^*, h(s,u(s))\right)d\mathrm{W}(s),\end{aligned}$$

And so

$$\begin{aligned}\left|u(t)-u^*\right|^2 &= \left|u(0)-u^*\right|^2 - 2\int_0^t \nu\left\|u(s)-u^*\right\|^2 ds \\ &\quad - 2\int_0^t b\left(u(s)-u^*, u^*, u(s)-u^*\right)ds \\ &\quad - 2\int_0^t \nu\left(C\left(u(s)-u^*\right), u(s)-u^*\right)_g ds \\ &\quad + \int_0^t \sigma^2\left|u(s)-u^*\right|^2 ds + 2\int_0^t \sigma\left|u(s)-u^*\right|^2 d\mathrm{W}(s).\end{aligned}$$

Now, Ito's formula for $\log\left|u(t)-u^*\right|^2$ yields that.

$$\begin{aligned}&\log\left|u(t)-u^*\right|^2 \\ &= \log\left|u(0)-u^*\right|^2 + \int_0^t \frac{2}{\left|u(t)-u^*\right|^2}\left[-\nu\left\|u(s)-u^*\right\|^2\right]ds \\ &\quad + \int_0^t \frac{2}{\left|u(t)-u^*\right|^2}\left[-\nu\left(C\left(u(s)-u^*\right), u(s)-u^*\right)_g\right]ds \\ &\quad + \int_0^t \frac{2}{\left|u(t)-u^*\right|^2}\left[-b\left(u(s)-u^*, u^*, u(s)-u^*\right)\right]ds \\ &\quad + \int_0^t \frac{\sigma^2\left|u(s)-u^*\right|^2}{\left|u(t)-u^*\right|^2}ds - \frac{1}{2}\int_0^t \frac{4\sigma^2\left|u(s)-u^*\right|^4}{\left|u(t)-u^*\right|^4}ds \\ &\quad + \int_0^t \frac{\sigma\left|u(s)-u^*\right|^2}{\left|u(t)-u^*\right|^2}d\mathrm{W}(s)\end{aligned}$$

Using Lemmas 2.1 and 2.2 we get.

$$\begin{aligned}&-\nu\left\|u(s)-u^*\right\|^2 - \nu\left(C\left(u(s)-u^*\right), u(s)-u^*\right)_g \\ &-b\left(u(s)-u^*, u^*, u(s)-u^*\right) \\ &\le \left(-\nu + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac{1}{2}}} + \frac{c_1}{\lambda_1^{\frac{1}{2}}}\left\|u^*\right\|^2\right)\left\|u(s)-u^*\right\|^2 \\ &\le \lambda_1\left[-\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac{1}{2}}}\right) + \frac{c_1}{\lambda_1^{\frac{1}{2}}}\left\|u^*\right\|^2\right]\left|u(s)-u^*\right|^2,\end{aligned}$$

Thanks to condition (4).

$$\begin{aligned}&\log\left|u(t)-u^*\right|^2 \\ &= \log\left|u(0)-u^*\right|^2 \\ &\quad + \lambda_1\left[-2\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac{1}{2}}}\right) + \frac{2c_1}{\lambda_1^{\frac{1}{2}}}\left\|u^*\right\|^2 - \frac{\sigma^2}{\lambda_1}\right]t + 2\sigma\mathrm{W}(t).\end{aligned}$$

Therefore, Letting $\lim_{t\to\infty}\frac{W(t)}{t}\to 0$ almost surely, we can find a set $\Omega_0 \subset \Omega$ with $P(\Omega_0)= 0$ such that, there exists $T(\omega)$ such that for all $\forall t \in T(\omega)$.

$$\frac{2\sigma W(t)}{t} \le -\frac{1}{2}\lambda_1\left[-2\nu\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac{1}{2}}}\right)+\frac{2c_1}{\lambda_1^{\frac{1}{2}}}\|u^*\|^2-\frac{\sigma^2}{\lambda_1}\right].$$

This deduces that.

Or, equivalently,

$$|u(t)-u^*|^2 \le |u_0-u^*|^2 e^{-\gamma t}, \forall t \ge T(\omega)$$

Where:

$$\gamma=\frac{1}{2}\lambda_1\left[2\nu\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac{1}{2}}}\right)-\frac{2c_1}{\lambda_1^{1/2}}\|u^*\|+\frac{\sigma^2}{\lambda_1}\right]>0.$$

The proof is complete.

REFERENCES

[1] C.T. Anh and D.T. Quyet (2012), *Long-time behavior for 2D non-autonomous g-Navier-Stokes equations*, Ann. Pol. Math, *103*, *277-302*.

[2] C.T. Anh, D.T. Quyet and D.T. Tinh (2013), *Existence and finite time approximation of strong solutions of the 2D g-Navier-Stokes equations*, Acta Math. Vietnam, 28, 413-428.

[3] H. Bae and J. Roh (2004), *Existence of solutions of the g-Navier-Stokes equations*, Taiwanese J. Math, 8, 85-102.

[4] J. Jiang and Y. Hou (2010), *Pullback attractor of 2D non-autonomous g-Navier-Stokes equations on some bounded domains*, App. Math. Mech. -Engl. Ed., 31, 697-708.

[5] J. Jiang and X. Wang (2013), *Global attractor of 2D autonomous g-Navier-Stokes equations*, Appl. Math. Mech. (English Ed.), 34, 385-394.

[6] M. Kwak, H. Kwean and J. Roh (2006), *The dimension of attractor of the 2D g-Navier-Stokes equations*, J. Math. Anal. Appl., 315, 436-461.

[7] H. Kwean and J. Roh (2005), *The global attractor of the 2D g-Navier-Stokes equations on some unbounded domains*, Commun. Korean Math. Soc., 20, 731-749.

[8] D.T. Quyet and N.V. Tuan (2017), *On the stationary solutions to 2D g-Navier-Stokes equations*, Acta Math. Vietnam., 42, 357-367.

[9] J. Roh (2005), *Dynamics of the g-Navier-Stokes equations*, J. Differential Equations, 211, 452-484.

[10] J. Roh (2006), *Derivation of the g-Navier-Stokes equations*, J. Chungcheon Math. Soc., 19, 213-218.

[11] N.V. Tuan, T.H. Yen (2019), *On the weak stationary solutions to 2D g-Navier-Stokes equations*, Scientific Journal SaoDo University., No 4 (67), 93-97.

[12] D. Wu and J. Tao (2012), *The exponential attractors for the g-Navier-Stokes equations*, J. Funct. Spaces Appl., Art. ID 503454, 12 pp.

THÔNG TIN VỀ TÁC GIẢ

Nguyen Viet Tuan

- Education

+ In 2006: Bachelor of Mathematics and Informatics, Hanoi Pedagogican University 2.

+ In 2011: Master of Mathematical, Hanoi Pedagogican University 2.

+ In 2019: PhD of Mathematics, Hanoi Pedagogican University 2.

- Working Experience

+ In 2007 to date: Lecturer at Faculty of Basic Sciences, Sao Do University.

- Research interests: Stability and stabilization for some evolution equations in fluid mechanics.

- Email: nguyentuandhsd@gmail.com.

- Phone: 0978235234.